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Discussion Papers in
Economics and Econometrics

Syntactic Foundations for Unawareness of Theorems

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No. 1002

ISSN 0966-4246

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January 26, 2010

Abstract

We provide a syntactic model of unawareness. By introducing multiple knowledge modalities, one for each sub-language, we specifically model agents whose only mistake in reasoning (other than their unawareness) is to underestimate the knowledge of more aware agents. We show that the model is a complete and sound axiomatization of the set-theoretic model of Galanis [2007] and compare it with other unawareness models in the literature.

JEL-Classifications: C70, C72, D80, D82.

Keywords: unawareness, uncertainty, knowledge, interactive epistemology, modal logic, bounded perception.

1 Introduction

An agent who is unaware lacks not only information, that is, answers to his questions, but, more importantly, he has a limited perception of the uncertainty he faces. In other words, he completely misses some of the relevant questions. This implies that there may be some pieces of information that the agent does not know, and at the same he does not know that he does not know. A less obvious implication of unawareness is that if the answers to questions are connected through theorems, then being unaware of these theorems implies that the knowledge of the other agents may be underestimated.

A short example illustrates the last point. Suppose that agent j is only aware of the statement "the price is high" and that the price will be determined tomorrow. In other words, it is simply not possible for anyone to know whether prices are high or low, if he is only aware of prices. Hence, agent j concludes that, in his awareness, i does not know whether the price is high. But suppose that agent i is also aware of the statement "the interest rate is low" and that there is a logical connection specifying that a low interest rate implies a high price. Agent i concludes that, in his awareness, he knows that the price is high. Although agent j is wrong when reasoning about i's knowledge, this is only because he is constrained by his unawareness. Because it is correct that no one can know whether the price is high if he is unaware of the interest rate, j is not making a mistake within the

*I am grateful to seminar participants at the 2009 Conference on the Theoretical Aspects of Rationality and Knowledge (TARK XII) at Stanford.

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bounds of his awareness.[1] Moreover, agent j is not irrational in general, that is, he does not make any other mistakes in his reasoning.

The paper provides a syntactic model of unawareness that isolates and captures this mistake of reasoning. Moreover, we show that it is a complete and sound axiomatization of the set-theoretic model of Galanis [2007]. The approach we use follows that of Heifetz et al. [2008a] (HMS from now on), of constructing a canonical unawareness structure. Finally, we compare the present model with that of HMS.

Although set-theoretic models about knowledge are prevalent in economics, syntactic models are in fact more transparent, both in terms of the assumptions they make and in terms of specifying the beliefs of the agents.[2] As a result, it is more straightforward to compare two approaches by looking at their syntactic representations, rather than their set-theoretic ones. In fact, such a comparison has already been made for most of the papers in the literature. By comparing the present model with that of HMS, we are able to determine its place in the literature.

In order to illustrate the difference between the present and other approaches, we need to distinguish between a language and a sub-language. When modeling knowledge using a syntactic approach, the modeler starts with a set of primitive propositions, consisting of statements like "it rains" or "the price is high". Using negation ($\neg$), conjunction ($\wedge$) and the knowledge modality k^i, a language is created, containing all the well formed formulas. Moreover, it is implicitly assumed that all agents have a perfect understanding of that language. For example, the formula "agent i knows that it rains" is equally understood by everyone. However, if we introduce unawareness, this may not be true.

Modica and Rustichini [1999] and HMS specify that apart from the universal language that is generated from all primitive propositions, there are also several sub-languages, each generated by *some* of the primitive propositions. An agent who is aware only of some primitive propositions describes the world using one sub-language, which may be very different from the sub-language used by another agent. Moreover, the agents may not comprehend or be unaware of other sub-languages.

Suppose there are two agents, each using a different sub-language, both containing the statement "the price is high". It is clear that both agents understand "prices" in the same way. For example, they can write contracts or bet on prices. However, does this imply that the statement "agent i knows that the price is high" is also understood in the same way by both agents? In other words, is knowledge when described in one sub-language identical to knowledge when described in another sub-language? In HMS and other papers in the literature the answer is "yes", so there is only one, objective, knowledge modality. In Galanis [2007] and in this paper we allow for the knowledge modality to be different across sub-languages. This captures the idea that agents of different perception (awareness) may reason differently about the knowledge of others.

One way of modeling this mistake in reasoning and the example, within the standard setting of a unique knowledge modality, is to drop the truth axiom, ($k^j\phi \rightarrow \phi$), which says that if j knows something then it is true. However, in this way we allow agents to be totally

[1] Agent j could also reason that i knows about prices just because he is smarter or more aware than him. Although this is allowed by the model, j cannot always make this claim, because then he will never be able to say that i *does not* know something. Hence, we also want to allow and capture the case where j mistakenly deduces that i does not know.

[2] See Aumann [1999] and Fagin et al. [1995] for a comparison of the two approaches.

irrational and to make all kinds of mistakes, even unrelated to unawareness.

In order to avoid this extra irrationality we introduce one knowledge modality, k^i_α, for each sub-language which is generated by a set α of primitive propositions. Moreover, we impose the truth axiom for each knowledge modality ($k^i_\alpha\phi \to \phi$) and we add an axiom saying that more complete sub-languages give a better description of knowledge. Formally, if $\alpha \subseteq \alpha'$ then $k^i_\alpha\phi \to k^i_{\alpha'}\phi$. Therefore, agent j can make a mistake about i's knowledge only if his sub-language is not more complete than i's sub-language. For example, we can simultaneously have $\neg k^i_\alpha\phi$ and $k^i_{\alpha'}\phi$ only if α' is not a subset of α.[3] Moreover, since the truth axiom holds for every sub-language, this is the only mistake in reasoning that any agent is allowed to make.

For instance, suppose j knows ϕ, so that $k^j_\alpha\phi$ is true. If ϕ is the statement "it rains" then it is true that it rains. But if ϕ is the statement $\neg k^i_\alpha\phi'$, then although $\neg k^i_\alpha\phi'$ is also true, it may be that because agent i's sub-language is α' (and $\alpha \subset \alpha'$) we also have that $k^i_{\alpha'}\phi'$ is true. Hence, i knows ϕ' and j essentially makes a mistake about i's knowledge. Allowing both $k^i_{\alpha'}\phi'$ and $k^i_\alpha\phi'$ to be true refers to the case where it is not possible to know ϕ' when being aware only of sub-language α, whereas it is possible to know ϕ' when being aware of the richer sub-language α'.

Summarizing, although one could argue that an "awareness leads to knowledge" effect is better captured in a dynamic environment, it has significant implications in a static model as well. As was described above, if two agents differ only in that one is more aware than the other, then the more aware agent would have more knowledge. Moreover, the less aware agent would mistakenly think that their knowledge is the same. This can be captured by relaxing the truth axiom in the standard model but then all possible mistakes (even unrelated to unawareness) are allowed. The other possibility is introducing multiple knowledge modalities, as described above.

A few clarifications are in order. First, since there are many knowledge modalities, which is the one that provides the true description of the agent's knowledge? This depends on the agent's sub-language, which is determined by his awareness. Consequently, when agent i reasons about j's knowledge, he first has to reason about j's awareness and sub-language.

Second, we do not allow for false certainties. In other words, it is never the case that an agent knows a formula which is false. This is due to the truth axiom. At the same time, we allow agents to make statements which, from another agent's point of view with a richer awareness, are mistaken. We say that agent i makes a mistake in his reasoning about j if, for example, he is aware only of primitive propositions in α and knows that j is aware of α, he knows that $\neg k^j_\alpha\phi$ and yet it is true that $k^j_{\alpha'}\phi$ and agent j is aware of all propositions in α', where $\alpha \subset \alpha'$. Because $\neg k^j_\alpha\phi$ is also true, the truth axiom is not violated. Moreover, i is not making a mistake when reasoning that j's awareness is α, when in fact it is α'. The reason is that i is only aware of primitive propositions in α, so he cannot reason above that level.

Comparing the present axiom system with that of HMS, we find two main differences. First, whereas in HMS knowledge in one sub-language is equivalent to knowledge in any other sub-language, here it only implies knowledge in more complete sub-languages.[4] Sec-

[3]In the example, i's sub-language is generated by primitive propositions in α' and j's sub-language is generated by α.

[4]The syntax of the two papers is not same. However, we are able to map the syntax of HMS to the syntax of the present paper in a natural way, so that the comparison of the axioms is meaningful.

ond, because in the present paper knowledge differs across sub-languages, the knowledge modalities "carry" awareness. For example, being aware of formula $k^i_\alpha\phi$ implies awareness of all propositions in α and is not equivalent to being aware of $k^i_\beta\phi$. This is not true in HMS, because there is only one knowledge modality. Hence, adapted to the syntax of the present paper, the axiom system of HMS specifies that awareness of $k^i_\alpha\phi$ only implies awareness of all primitive propositions that generate ϕ, and it is equivalent to awareness of $k^i_\beta\phi$. This second difference implies, as we show in the following section, that the axiom system of HMS is neither weaker nor stronger than the axiom system of this paper.

Fagin and Halpern [1988] provide the first model of unawareness and introduce an explicit awareness operator, as is the case with the present paper. Modica and Rustichini [1994, 1999], Dekel et al. [1998] and HMS define awareness in terms of knowledge. Both HMS and Halpern and Rêgo [2008] provide sound and complete axiomatizations of Heifetz et al. [2006], hence they are equivalent. Moreover, they are multi agent generalizations of Modica and Rustichini [1999] and Halpern [2001], respectively, which are also equivalent. HMS is also equivalent to a multi agent version of a sub-class of unawareness structures described in Fagin and Halpern [1988].[5] Board and Chung [2007] provide a model of unawareness using first order modal logic.

Heifetz et al. [2006], Li [2008] and Galanis [2007] construct set-theoretic models of unawareness using multiple state spaces. On the other hand, Geanakoplos [1989], Ely [1998] and Xiong [2007] employ the standard framework of a unique state space. Dekel et al. [1998] argue that if unawareness satisfies three plausible properties, then a standard state space can only accommodate trivial unawareness.

Games with unawareness are analyzed by Feinberg [2004, 2005], Čopič and Galeotti [2007], Li [2006b], Sadzik [2006], Heifetz et al. [2007], Heifetz et al. [2008b] and Halpern and Rêgo [2006]. Applications with unawareness have been provided by Modica et al. [1998], Ewerhart [2001], Galanis [2008], Filiz-Ozbay [2008], Ozbay [2008], von Thadden and Zhao [2008] and Zhao [2008].

The paper is organized as follows. Section 2 presents the syntax and the axiom system and compares it to that of HMS. In section 3 we define the unawareness structures and in section 4 we construct the canonical structure. Soundness and completeness are demonstrated in section 5. All proofs are included in the appendix.

2 Syntax and axiom system

Let X be the set of primitive propositions, and let I be the set of individuals. The syntax we use involves the usual modalities $\neg$, $\wedge$ and the unusual modalities k^i_α and a^i_α, where $\alpha \subseteq X$. That is, instead of the "objective" knowledge and awareness modalities k^i and a^i, we introduce one for each subset α of the set of primitive propositions.

Given a sequence of primitive propositions and modalities, ϕ, let $Pr(\phi)$ be the set of primitive propositions contained in ϕ.[6] More precisely,

- $Pr(\top) = \emptyset$,

[5] For more details on the relationships between these papers, see HMS and Halpern and Rêgo [2008].

[6] The definition of Pr suggests that modalities like k^i_α and a^i_α are also considered "primitive propositions". For example, we can have $Pr(k^i_\alpha\phi) = \alpha \supsetneq Pr(\phi)$. HMS also define a Pr function, but their definition is different. We elaborate on the differences in the next section.

- $Pr(x) = \{x\}$, for $x \in X$,
- $Pr(\neg\phi) = Pr(\phi)$,
- $Pr(\phi \wedge \psi) = Pr(\phi) \cup Pr(\psi)$,
- $Pr(k^i_\alpha \phi) = Pr(\phi) \cup \alpha$,
- $Pr(a^i_\alpha \phi) = Pr(\phi) \cup \alpha$.

The set of formulas $\mathcal{L}$ is the smallest set such that:

- $\top$ is a formula,
- every $x \in X$ is a formula,
- if ϕ is a formula, then $\neg\phi$ is a formula,
- if ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula,
- if ϕ is a formula and $Pr(\phi) \subseteq \alpha \subseteq X$, then $a^i_\alpha \phi$ and $k^i_\alpha \phi$ are formulas.

Call $\mathcal{L}$ the "universal" language. Given a subset $\alpha \in X$, define the sub-language $\mathcal{L}_\alpha := \{\phi \in \mathcal{L} : Pr(\phi) \subseteq \alpha\}$, which consists of the formulas and the knowledge and awareness modalities containing only primitive propositions in α.

Consider the following axiom system.

- All substitution instances of valid formulas of Propositional Calculus including the formula $\top$, (PC),
- the inference rule *Modus Ponens*:
$$\frac{\phi, \phi \to \psi}{\psi}, \tag{MP}$$

  For $Pr(\phi), Pr(\psi) \subseteq \beta \subseteq \alpha \subseteq X$,

- the *Axiom of Truth*:
$$k^i_\alpha \phi \to \phi, \tag{T}$$
- the *Axiom of Positive Introspection*:
$$k^i_\alpha \phi \bigwedge_{x\in\beta} a^i_\alpha x \bigwedge_{y\in\alpha\setminus\beta} \neg a^i_\alpha y \to k^i_\alpha k^i_\beta \phi, \tag{4}$$
- the *Axiom of Negative Introspection* :
$$a^i_\alpha \phi \bigwedge_{x\in\beta} a^i_\alpha x \to k^i_\alpha \phi \vee k^i_\alpha(\neg k^i_\beta \phi \wedge a^i_\beta \phi), \tag{5}$$
- the *Propositional Awareness* Axioms:
$$\begin{array}{ll} 1. & a^i_\alpha \phi \leftrightarrow a^i_\alpha \neg\phi, \\ 2. & a^i_\alpha \phi \wedge a^i_\alpha \psi \leftrightarrow a^i_\alpha(\phi \wedge \psi), \\ 3. & a^i_\alpha k^j_\beta \phi \leftrightarrow \bigwedge_{x\in\beta} a^i_\alpha x, \text{ for } j \in I, \\ 4. & a^i_\alpha a^j_\beta \phi \leftrightarrow \bigwedge_{x\in\beta} a^i_\alpha x, \text{ for } j \in I. \end{array} \tag{PA}$$

- the inference rule *RK-Inference*: For all natural numbers $n \geq 1$: If $\phi_1, \phi_2, \ldots, \phi_n$ and ϕ are formulas such that $Pr(\phi) \subseteq U_{i=1}^{n} Pr(\phi_i) \subseteq \alpha \subseteq X$ then

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n \to \phi}{k_\alpha^i \phi_1 \wedge \ldots \wedge k_\alpha^i \phi_n \to k_\alpha^i \phi}. \tag{RK}$$

- For $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$,

$$k_\alpha^i \phi \to a_\alpha^i \phi, \tag{A}$$

$$a_\alpha^i \phi \leftrightarrow a_{\alpha'}^i \phi, \tag{AA}$$

$$k_\alpha^i \phi \to k_{\alpha'}^i \phi. \tag{KA}$$

Axioms PC and MP are standard and need no explanation. Axioms T,4 and 5 are adapted versions of the following familiar axioms:

$$k^i \phi \to \phi,$$

$$k^i \phi \to k^i k^i \phi,$$

$$k^i \phi \vee k^i \neg k^i \phi.$$

The main difference is that these axioms are expressed in a syntax with more than one knowledge modality. Hence, T says that the truth axiom holds for all knowledge modalities. Axiom 4 says that if, within the sub-language generated by primitive propositions in α, the agent knows ϕ and he is aware only of primitive propositions in β, then he knows that he knows ϕ, where "he knows ϕ" is expressed in the sub-language generated by β. Note that the sub-language generated by α cannot express awareness of a primitive proposition outside α. Axiom 5 specifies that being aware of ϕ and all primitive propositions in β implies that either he knows ϕ, or that he knows that, within the sub-language generated by β, he does not know it and he is aware of it.

Axioms PA1 and PA2 are used in Modica and Rustichini [1999] and in HMS but here they are extended for all awareness modalities of all sub-languages. Axiom PA3 specifies that agent i is aware that, within the sub-language generated by β, agent j knows ϕ, if and only if i is aware of all primitive propositions in β. This Axiom is similar to the PA3 Axiom of HMS: $a^i \phi \leftrightarrow a^i k^j \phi$, for $j \in I$. However, as we discuss in the following section, neither is weaker or stronger than the other. Axiom PA4 has similar intuition.[7] RK-Inference is similar to the RK-Inference rule introduced by HMS. There are two differences. First, the Pr function here is different from the Pr function in HMS. We elaborate on this difference in the following section. Second, the rule here applies to all permitted knowledge modalities. Axiom A specifies that knowledge implies awareness.

The last two Axioms specify when awareness and knowledge in one sub-language translate to awareness and knowledge to another sub-language. Axiom AA says that awareness of a formula ϕ in a sub-language generated by α implies awareness of ϕ in all sub-languages which are either more or less complete and can express ϕ. Axiom KA specifies that knowledge of ϕ in a sub-language generated by α implies knowledge of ϕ only in sub-languages which are more complete. This last axiom essentially relaxes the condition that there is one, objective,

[7]HMS do not have a PA4 Axiom, as they define awareness as $a^i \phi := k^i \phi \vee k^i \neg k^i \phi$, whereas here the only connection between the awareness and knowledge modalities is through the axioms.

knowledge modality, that transcends all sub-languages, as in HMS and other papers in the literature.

The following definitions are standard and taken from HMS.

Definition 1. *The set of theorems is the smallest set of formulas that contain all the axioms and that is closed under the inference rules Modus Ponens and RK-Inference.*

Definition 2. *Let Γ be a set of formulas and ϕ a formula. A proof of ϕ from Γ is a finite sequence of formulas such that the last formula is ϕ and such that each formula is a formula in Γ, a theorem of the system or inferred from the previous formulas by Modus Ponens. If there is a proof of ϕ from Γ, then we write $\Gamma \vdash \phi$. In particular, $\vdash \phi$ means that ϕ is a theorem. If $\Gamma \vdash \phi$, we say that Γ implies ϕ syntactically.*

Definition 3. *A set of formulas is consistent if and only if there is no formula ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. A set Γ of formulas is inconsistent, if it is not consistent.*

2.1 Relation to the axiom system of HMS

In this section we compare the present axiom system with that of HMS. The main difficulty is that the syntax of the two approaches is different. In particular, whereas HMS have one knowledge modality k^i and one awareness modality a^i, the syntax of the present paper contains several knowledge and awareness modalities, k^i_α, a^i_α, one for each subset $\alpha \subseteq X$ of primitive propositions.

We can only have a meaningful comparison if the syntax is the same. This can be achieved if we interpret k^i, a^i in the HMS syntax as the modalities k^i_X, a^i_X, respectively, in the syntax of this paper, where X is the set of all primitive propositions. Moreover, we add to the axiom system of HMS two axioms specifying that all knowledge and awareness modalities are the "same". That is, if $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$, we have $k^i_\alpha\phi \leftrightarrow k^i_{\alpha'}\phi$ and $a^i_\alpha\phi \leftrightarrow a^i_{\alpha'}\phi$. HMS define the awareness modality as $a^i\phi := k^i\phi \vee k^i\neg k^i\phi$. We incorporate this definition as an axiom in their axiom system.

Finally, the definition of the function Pr in HMS is different from the definition here. Adapted to the syntax of the present paper, Pr in HMS requires that $Pr(k_\alpha\phi) = Pr(\phi)$, whereas here it requires that $Pr(k_\alpha\phi) = Pr(\phi) \cup \alpha$. This difference matters for the definition of RK-Inference. To distinguish between the two, we denote as Pr' the function Pr of HMS.

Summarizing, the HMS axiom system, adapted to the syntax of the present paper and with the addition of the aforementioned axioms, takes the following form. We denote similar axioms with a $'$.

- Axioms (PC), (MP),
- the *Axiom of Truth*:
$$k^i_X\phi \to \phi, \tag{T$'$}$$
- the *Axiom of Positive Introspection*:
$$k^i_X\phi \to k^i_X k^i_X\phi,, \tag{4$'$}$$

- the *Propositional Awareness* Axioms:

$$\begin{array}{l} 1.\ a^i_X\phi \leftrightarrow a^i_X\neg\phi, \\ 2.\ a^i_X\phi \wedge a^i_X\psi \leftrightarrow a^i_X(\phi \wedge \psi), \\ 3.\ a^i_X\phi \leftrightarrow a^i_X k^j_X\phi, \text{ for } j \in I. \end{array} \tag{PA$'$}$$

- the inference rule *RK-Inference*: For all natural numbers $n \geq 1$: If $\phi_1, \phi_2, \ldots, \phi_n$ and ϕ are formulas such that $Pr'(\phi) \subseteq U^n_{i=1}Pr'(\phi_i)$ then

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \phi}{k^i_X\phi_1 \wedge \ldots \wedge k^i_X\phi_n \rightarrow k^i_X\phi}. \tag{RK$'$}$$

For $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$,[8]

$$a^i_\alpha\phi \leftrightarrow a^i_{\alpha'}\phi, \tag{AA}$$

$$k^i_\alpha\phi \leftrightarrow k^i_{\alpha'}\phi, \tag{KA$'$}$$

$$a^i_X\phi \leftrightarrow k^i_X\phi \vee k^i_X\neg k^i_X\phi. \tag{D}$$

The first difference between the two axiom systems is that KA$'$ is relaxed to KA. That is, whereas in HMS there is effectively only one knowledge operator that transcends all sub-languages, in the present axiom system knowledge in one sub-language only implies knowledge in more complete sub-languages.

The second difference is that in the HMS system knowledge and awareness modalities k^i_α and a^i_α do not "carry" any awareness. It is a theorem of the HMS system that being aware of the formula $k^i_\alpha\phi$ is equivalent to being aware of formula $k^i_{\alpha'}\phi$, for any $\alpha' \subseteq X$.[9] This is consistent with the premise that there is effectively only one knowledge modality, k^i. In contrast, in the approach of the present paper knowledge operators "carry" awareness, so that being aware of formula $k^i_\alpha\phi$ does not imply awareness of $k^i_{\alpha'}\phi$. The difference between the two approaches is illustrated by Axioms PA3 and PA$'$3. Although they look similar, it is not the case that one is weaker than the other.

In particular, PA$'$3 is not a theorem of the current axiom system. To see this, note that if this were the case, then $a^i_X\phi \rightarrow a^i_X k^j_X\phi$ and PA3 would imply that whenever i is aware of a formula ϕ, he is also aware of all primitive propositions in X. For the same reason, PA3 and PA4 are not theorems of the HMS axiom system.[10] As a result, it is not the case that the present axiom system is either weaker or stronger than the HMS axiom system.

The following proposition shows that the remaining axioms are theorems of the HMS system. Let inference rule RK$''$ be the same as RK but adding the qualification that $Pr'(\phi) \subseteq U^n_{i=1}Pr'(\phi_i)$.

Proposition 1. *Axioms PC, T, 4, 5, PA1, PA2, A, AA, KA and inference rules MP and RK$''$ are derived from the axiom system of HMS.*

[8] Note that we use the Pr function, not the Pr' one, because we want the two axioms to hold for all knowledge and awareness modalities that can express formula ϕ.

[9] This is derived from the Propositional Awareness Axioms, Lemma 1 in HMS, and the definition of Pr'.

[10] Another difference is between inference rules RK and RK$'$, because Pr is not equivalent to Pr'. Pr specifies that k^i_α, a^i_α carry awareness, so that $Pr(k^i_\alpha\phi) = Pr(\phi) \cup \alpha$, whereas Pr' specifies that they do not, so that $Pr(k^i_\alpha\phi) = Pr(\phi)$.

3 Unawareness structures

We first present an overview of the model developed in Galanis [2007]. Consider a complete lattice of disjoint state spaces $\mathcal{S} = \{S_a\}_{a \in A}$ and denote by $\Sigma = \cup_{a \in A} S_a$ the union of these state spaces. A state ω is an element of some state space S. Let S^* be the most complete state space, the join of all state spaces in $\mathcal{S}$. We call S^* the full state space. An element $\omega^* \in S^*$ is called a full state.

Let $\preceq$ be a partial order on $\mathcal{S}$. For any $S, S' \in \mathcal{S}$, $S \preceq S'$ means that S' is more expressive than S. Moreover, there is a surjective projection $r_S^{S'} : S' \to S$. Projections are required to commute. If $S \preceq S' \preceq S''$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$ and $\omega_{S''} = \{\omega' \in S'' : r_{S'}^{S''}(\omega') = \omega\}$. If $B \subseteq S'$, denote by $B_S = \{\omega_S : \omega \in B\}$ the restriction of B on S and by $B_{S''} = \bigcup\{\omega_{S''} : \omega \in B\}$ the enlargement of B on S''. Let $g(S) = \{S' : S \preceq S'\}$ be the collection of state spaces that are at least as expressive as S. For a set $B \subseteq S$, denote by $B^\uparrow = \bigcup_{S' \in g(S)} B_{S'}$ the enlargements of B to all state spaces which are at least as expressive as S.

Consider a possibility correspondence $P^i : \Sigma \to 2^\Sigma \setminus \emptyset$ with the following properties:

(0) Confinedness: If $\omega \in S$ then $P^i(\omega) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $\omega \in (P^i(\omega))^\uparrow$ for every $\omega \in \Sigma$.

(2) Stationarity: $\omega' \in P^i(\omega)$ implies $P^i(\omega') = P^i(\omega)$.

(3) Projections Preserve Awareness: If $\omega \in S'$, $\omega \in P^i(\omega)$ and $S \preceq S'$ then $\omega_S \in P^i(\omega_S)$.

(4) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $(P^i(\omega))^\uparrow \subseteq (P^i(\omega_S))^\uparrow$.

The setting is the same with that of Heifetz et al. [2006]. The first difference is that we completely take out their Axiom Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $P^i(\omega) \subseteq S'$ then $(P^i(\omega))_S = P^i(\omega_S)$. Justification and examples for this omission are provided in Galanis [2007]. The two other differences concern the definitions of an event and those of knowledge and awareness.

3.1 Events, awareness and knowledge

Formally, an *event* is a pair (E, S), where $E \subseteq S$ and $S \in \mathcal{S}$. The negation of (E, S), defined by $\neg(E, S) = (S \setminus E, S)$, is the complement of E with respect to S. Let $\mathcal{E} = \{(E, S) : E \subseteq S, S \in \mathcal{S}\}$ be the set of all events. We write E as a shorthand for (E, S) and $\emptyset_S$ as a shorthand for $(\emptyset, S)$. For each event E, let $S(E)$ be the state space of which it is a subset. An event E "inherits" the expressiveness of the state space of which it is a subset. Hence, we can extend $\preceq$ to a partial order $\preceq_0$ on $\mathcal{E}$ in the following way: $E \preceq_0 E'$ if and only if $S(E) \preceq S(E')$. Abusing notation, we write $\preceq$ instead of $\preceq_0$.

Before defining knowledge, we need to define awareness. For any event E, for any state space S such that $S \succeq E$, define

$$A_S^i(E) = \{\omega \in S : E \preceq P^i(\omega)\}$$

to be the event which describes, with the vocabulary of S, that the agent is aware of event E. The agent is aware of an event whenever his possibility set resides in a state space that is rich enough to express event E. Unawareness is defined as the negation of awareness. More formally, the event $U^i_S(E)$ describes, with the vocabulary of S, that the agent is unaware of E:

$$U^i_S(E) = \neg A^i_S(E) = (S \setminus A^i_S(E), S).$$

Let $\Omega^i : \Sigma \to \mathcal{S}$ be such that for any $\omega \in \Sigma$, $\Omega^i(\omega) = S$ if and only if $P^i(\omega) \subseteq S$. $\Omega^i(\omega)$ denotes the agent's state space at ω. An agent knows an event E if he is aware of it and in all the states he considers possible, E is true. Formally, for any event E and for any state space S such that $S \succeq E$, define

$$K^i_S(E) = \{\omega \in A^i_S(E) : P^i(\omega) \subseteq E_{\Omega^i(\omega)}\}.$$

An unawareness structure is defined to be, as in HMS, the tuple

$$\underline{\Sigma} = \left\langle\ (S_\alpha)_{\alpha\in A}\, , \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, \left(P^i\right)_{i\in I} \right\rangle.$$

4 The canonical structure

Recall that, given a subset $\alpha \in X$, $\mathcal{L}_\alpha = \{\phi \in \mathcal{L} : Pr(\phi) \subseteq \alpha\}$ is the sub-language generated by the set α of primitive propositions. Given $\alpha \subseteq X$, define Ω_α to be the set of maximally consistent sets ω_α of formulas in $\mathcal{L}_\alpha$. Let $\Omega = \cup_{\alpha\subseteq X}\Omega_\alpha$ be the collection of all state spaces and define $\Omega_\beta \preceq \Omega_\alpha$ whenever $\beta \subseteq \alpha$. If $\Omega_\beta \preceq \Omega_\alpha$ then the projection $r^\alpha_\beta : \Omega_\alpha \to \Omega_\beta$ is defined as $r^\alpha_\beta(\omega) := \omega \cap \mathcal{L}_\beta$. From Proposition 3 and Remark 2 of HMS, the projection r^α_β is well defined and surjective, and $\alpha \supseteq \beta \supseteq \gamma$ implies $r^\alpha_\gamma = r^\beta_\gamma \circ r^\alpha_\beta$.

Given a formula ϕ and a subset $\alpha \supseteq Pr(\phi)$, $[\phi]_{\Omega_\alpha} := \{\omega \in \Omega_\alpha : \phi \in \omega\}$ is an event, as it is a subset of state space Ω_α.

Definition 4. *For $\omega \in \Omega_\alpha$, $\alpha \subseteq X$ and $i \in I$, set*

$$P^i(\omega) := \left\{\omega' \in \Omega : \textit{For every formula } \phi \begin{array}{ll} i)\ k^i_\alpha\phi \in \omega & \textit{implies } \phi \in \omega' \\ ii)\ a^i_\alpha\phi \in \omega & \textit{iff } (\phi \in \omega' \textit{ or } \neg\phi \in \omega') \end{array} \right\}.$$ [11]

Proposition 2. *For every $i \in I$ and $\omega \in \Sigma$, $P^i(\omega)$ is nonempty and satisfies properties 0-4.*

Corollary 1. *The tuple*

$$\underline{\Omega} = \left\langle\ (\Omega_\alpha)_{\alpha\subseteq X}\, , \left(r^\alpha_\beta\right)_{\beta\subseteq\alpha\subseteq X}, \left(P^i\right)_{i\in I} \right\rangle,$$

is an unawareness structure.

Moreover, as the following lemma shows, knowledge and awareness can interchangeably be described syntactically or as an event.

Lemma 1. *Suppose that $\phi \in \mathcal{L}$ and $Pr(\phi) \subseteq \beta \subseteq \alpha \subseteq X$. Then,*

$$[\neg\phi]_{\Omega_\alpha} = \neg[\phi]_{\Omega_\alpha},$$

[11] Note that $k^i_\alpha\phi, a^i_\alpha\phi$ are defined only if $Pr(\phi) \subseteq \alpha$.

$$[\phi \wedge \psi]_{\Omega_\alpha} = [\phi]_{\Omega_\alpha} \cap [\psi]_{\Omega_\alpha},$$

$$[k^i_\beta \phi]_{\Omega_\alpha} = \left(K^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}}) \right)_{\Omega_\alpha},$$

$$[a^i_\beta \phi]_{\Omega_\alpha} = \left(A^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}}) \right)_{\Omega_\alpha}.$$

Given a formula ϕ, a state $\omega \in \Omega_\alpha$ contains a sequence of knowledge modalities $k^i_{\alpha'}\phi$, where α' is such that $Pr(\phi) \subseteq \alpha' \subseteq \alpha$. Which one of the knowledge modalities is the "true" description of i's knowledge of ϕ? This depends on i's awareness. If ω specifies that i is aware only of primitive propositions in $\alpha' \subseteq \alpha$, then his sub-language is $\Omega_{\alpha'}$ and he knows ϕ if $k^i_{\alpha'}\phi \in \omega$.

It is important to stress that Ω_α, as a description of i's knowledge, can be quite restrictive. The reason is that agent i may be aware of a primitive proposition x which does not belong to α. As a result, sub-language Ω_α is not complete enough to express awareness of x. But more importantly, in that case Ω_α is also not complete enough to express i's knowledge of ϕ as well. In particular, if $\alpha'' = \alpha \cup \{x\}$ then the modality $k^i_{\alpha''}\phi$ is better suited to describe i's knowledge. But it does not belong to the sub-language $\mathcal{L}_\alpha$, so it is not part of any state in Ω_α.

According to the axiom system, a less complete sub-language can only underestimate one's knowledge, not overestimate it. In particular, suppose that $k^i_\alpha \phi \in \omega$ and $\underset{x\in\alpha}{\wedge} a^i_\alpha x \in \omega$ so that i knows ϕ, according to ω. Because of Axiom KA, it must be that $k^i_{\alpha''}\phi \in \omega'$ for any $\omega' \in \Omega_{\alpha''}$ that projects to ω, where $\alpha \subset \alpha''$. On the other hand, if $\neg k^i_\alpha \phi \in \omega$ we may have $k^i_{\alpha''}\phi \in \omega'$ or $\neg k^i_{\alpha''}\phi \in \omega'$. Hence, more complete state spaces give a better description of one's knowledge.

Summarizing, it may be that ω' specifies that agent i knows ϕ, whereas the projection of ω' to a lower state space specifies that he does not know ϕ. This is the Awareness Leads to Knowledge property, proposed in Galanis [2007]. The intuition behind this property is that the projection belongs to a state space which is generated by a less complete sub-language, hence containing fewer knowledge modalities $k^i_{\alpha'}\phi$, which may underestimate i's knowledge. This property is not true in HMS, effectively because there is only one knowledge modality, k^i.

5 Soundness and completeness

Recall that $\mathcal{E} = \{E \subseteq S : S \in \mathcal{S}\}$ is the collection of all events and let $\mathcal{E}^\uparrow := \{E^\uparrow : E \in \mathcal{E}\}$ be the collection of *extended events*. A typical element of $\mathcal{E}^\uparrow$ consists of an event $E \subseteq S$ and all of its enlargements to higher state spaces $E_{S'}$, where $S \preceq S'$. For a given set of primitive propositions X, let $v : X \to \mathcal{E}^\uparrow$ be the evaluation function. The extended event $v(x)$ contains all events where the primitive proposition x obtains. An unawareness model is a pair $\underline{\Sigma}^v := (\underline{\Sigma}, v)$. Abusing notation, we write $\underline{\Sigma}$ for an unawareness model, instead of $\underline{\Sigma}^v$. Let $C : \mathcal{S} \to 2^X$ denote which primitive propositions in X occur in state space S. That is, define, for each $S \in \mathcal{S}$, $C(S) := \bigcup\{x \in X : E \in v(x), E \subseteq S\}$. We assume that if $S \neq S'$ then $C(S) \neq C(S')$. Given any set $\alpha \subseteq X$, let $C^{-1}(\alpha) := \bigwedge\{S \in \mathcal{S} : \alpha \subseteq C(S)\}$ be the least complete state space where all primitive propositions in α occur.

We first specify what it means for a formula ϕ to be defined at a particular state ω.

Definition 5. *For a nonempty set X and a set of players I, let $(\underline{\Sigma}, v)$ be an unawareness model, and let $\omega \in S$ for some $S \in \mathcal{S}$. Then we define by induction on the formation of the formulas in $\mathcal{L}$:*

- $(\underline{\Sigma}, \omega) \mapsto \top$, *for all* $\omega \in \Sigma$,
- $(\underline{\Sigma}, \omega) \mapsto x$, *if* $\omega \in E \in E^{\uparrow} = v(x)$,
- $(\underline{\Sigma}, \omega) \mapsto \phi \wedge \psi$, *if* $(\underline{\Sigma}, \omega) \mapsto \phi$ *and* $(\underline{\Sigma}, \omega) \mapsto \psi$,
- $(\underline{\Sigma}, \omega) \mapsto \neg\phi$, *if* $(\underline{\Sigma}, \omega) \mapsto \phi$,
- $(\underline{\Sigma}, \omega) \mapsto a^i_\alpha \phi$, *if* $Pr(\phi) \subseteq \alpha = C(S')$, $S' \preceq S$, *and* $(\underline{\Sigma}, \omega) \mapsto \phi$,
- $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha \phi$, *if* $Pr(\phi) \subseteq \alpha = C(S')$, $S' \preceq S$, *and* $(\underline{\Sigma}, \omega) \mapsto \phi$.

Definition 6. *Say that a formula ϕ is defined at state $\omega \in S \in \mathcal{S}$ of unawareness model $\underline{\Sigma}$ if $(\underline{\Sigma}, \omega) \mapsto \phi$.*

Note that $k^i_\alpha \phi$, $a^i_\alpha \phi$ are defined at state $\omega \in S$ only if the set of primitive propositions α corresponds to a state space S' ($C(S') = \alpha$) that is less complete than S. In that way, we get a one to one correspondence between the knowledge (awareness) modality k^i_α (a^i_α) and the knowledge (awareness) operator $K^i_{C^{-1}(\alpha)}$ ($A^i_{C^{-1}(\alpha)}$). As the following definition shows, the negation of a formula is true if it is defined but not true.

Definition 7. *For a nonempty set X and a set of players I, let $(\underline{\Sigma}, v)$ be an unawareness model, and let $\omega \in S$ for some $S \in \mathcal{S}$. Then we define by induction on the formation of the formulas in $\mathcal{L}$:*

- $(\underline{\Sigma}, \omega) \models \top$, *for all* $\omega \in \Sigma$,
- $(\underline{\Sigma}, \omega) \models x$, *if* $\omega \in E \in E^{\uparrow} = v(x)$,
- $(\underline{\Sigma}, \omega) \models \phi \wedge \psi$, *if* $(\underline{\Sigma}, \omega) \models \phi$ *and* $(\underline{\Sigma}, \omega) \models \psi$,
- $(\underline{\Sigma}, \omega) \models \neg\phi$, *if* $(\underline{\Sigma}, \omega) \mapsto \neg\phi$ *and not* $(\underline{\Sigma}, \omega) \models \phi$,
- $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi$, *if* $(\underline{\Sigma}, \omega) \mapsto a^i_\alpha \phi$ *and* $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$,
- $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi$, *if* $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha \phi$ *and* $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$,

where, given a formula ψ and $S \in \mathcal{S}$ such that $(\underline{\Sigma}, \omega) \mapsto \psi$ for some $\omega \in S$, $[\psi]_S := \{\omega \in S : (\underline{\Sigma}, \omega) \models \psi\}$. Moreover, $\neg[\psi]_S$ is the complement with respect to S.

The following definitions are standard.

Definition 8. *We say that ϕ is true in state ω if $(\underline{\Sigma}, \omega) \models \phi$. For a set of formulas Γ, we say that Γ is true in state ω if $(\underline{\Sigma}, \omega) \models \phi$, for all $\phi \in \Gamma$.*

Definition 9. *For $\Gamma \subseteq \mathcal{L}$, we say that Γ has a model if there is an unawareness model $\underline{\Sigma}$ and a state $\omega \in \Sigma$ such that $(\underline{\Sigma}, \omega) \models \Gamma$.*

Definition 10. *Let $\underline{\Sigma}$ be an unawareness model. If Γ is a set of formulas and ϕ is a formula, we write $\Gamma \models_{\underline{\Sigma}} \phi$ if whenever ϕ is defined at state $\omega \in S$ we have that $(\underline{\Sigma}, \omega) \models \Gamma$ implies $(\underline{\Sigma}, \omega) \models \phi$.*

Definition 11. *We write $\Gamma \models \phi$ if for every unawareness model $\underline{\Sigma}$ we have $\Gamma \models_{\underline{\Sigma}} \phi$. In this case, we say that Γ implies ϕ semantically. Accordingly, we write $\models \phi$ if it is the case that $\emptyset \models \phi$. We say that ϕ is valid, if $\models \phi$.*

Definition 12. *The system of axioms and inference rules is strongly sound (with respect to the class of unawareness models) if for every set of formulas Γ and every formula ϕ we have that $\Gamma \vdash \phi$ implies $\Gamma \models \phi$. It is strongly complete if the reverse holds.*

Definition 13. *For $x \in X$, define $v^{\Omega}(x) := \{\omega \in \Omega : x \in \omega\}$.*

Corollary 2. *The pair $(\underline{\Omega}, v^{\Omega})$ is an unawareness model such that for all $\phi \in \mathcal{L}$:*

$$(\underline{\Omega}, \omega) \models \phi \textit{ iff } \phi \in \omega.$$

HMS call $(\underline{\Omega}, v^{\Omega})$ the canonical unawareness model. We use it to prove the following Theorem, which provides the syntactic foundations for the set-theoretic model of Galanis [2007].

Theorem 1. *The system of axioms is strongly sound and complete with respect to the class of unawareness models.*

A Appendix

We use the following inference rules: Conjunction,

$$\frac{\phi, \psi}{\phi \wedge \psi},$$

and Implication,

$$\frac{\phi \to \psi, \psi \to \chi}{\phi \to \chi}.$$

They are both derived from PC and MP. For details, see HMS.

Proof of Proposition 1. Axiom T is derived from KA′ and T′. For Axiom 4, note that $k^i_\alpha \phi \to k^i_X \phi$, $k^i_X \phi \to k^i_X k^i_X \phi$ and $k^i_X k^i_X \phi \to k^i_X k^i_\alpha \phi$ are theorems, because of Axioms KA′, 4′ and RK′. Similarly, because of KA′ and RK′ we have $k^i_X k^i_\alpha \phi \to k^i_X k^i_\beta \phi$ and $k^i_X k^i_\beta \phi \to k^i_\alpha k^i_\beta \phi$. By Implication we have $k^i_\alpha \phi \to k^i_\alpha k^i_\beta \phi$.

For Axiom 5, we have from Axioms D, AA, KA′ and inference rule RK′ that $a^i_\alpha \phi \to k^i_\alpha \phi \vee k^i_\alpha \neg k^i_\alpha \phi$ is a theorem. From RK′ and KA′ we have that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_X \neg k^i_\alpha \phi$, $k^i_X \neg k^i_\alpha \phi \to k^i_X \neg k^i_\beta \phi$ and $k^i_X \neg k^i_\beta \phi \to k^i_\alpha \neg k^i_\beta \phi$ are theorems. From Implication, we have that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha \neg k^i_\beta \phi$ is a theorem. We then show that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha a^i_\beta \phi$. Note first that $a^i_\beta \phi \leftrightarrow k^i_\beta \phi \vee k^i_\beta \neg k^i_\beta \phi$ is a theorem. It then suffices to show (from KA′ and RK′) that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha k^i_\beta \phi \vee k^i_\alpha k^i_\beta \neg k^i_\beta \phi$ is a theorem. Because $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha \neg k^i_\beta \phi$ is a theorem and from Axiom KA′ we have that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\beta \neg k^i_\beta \phi$ is a theorem. Hence, we also have, from RK′ and KA′, that $k^i_\alpha k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha k^i_\beta \neg k^i_\beta \phi$ is a theorem. From Axioms 4′, KA′ and RK′ we have that $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha k^i_\alpha \neg k^i_\alpha \phi$ is a theorem. From Implication, $k^i_\alpha \neg k^i_\alpha \phi \to k^i_\alpha k^i_\beta \neg k^i_\beta \phi$ is a theorem. Finally, $k^i_\alpha k^i_\beta \neg k^i_\beta \phi \to k^i_\alpha k^i_\beta \phi \vee k^i_\alpha k^i_\beta \neg k^i_\beta \phi$ is a theorem.

Axiom A is a theorem from Axioms KA′, AA and D. Axiom AA is also an axiom in HMS. For Axiom PA1 we have, because of Axioms AA, PA′1, that $a^i_\alpha \phi \leftrightarrow a^i_X \phi \leftrightarrow a^i_X \neg\phi \leftrightarrow a^i_{\alpha'} \neg\phi$. From Implication, we have the desired result. Similar logic applies for PA2. Axiom KA is derived from Axiom KA′. Inference rule RK″ is derived from RK′ and KA′.

□

Lemma 2. *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be nonempty sets of formulas, each closed under conjunctions, such that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is inconsistent. Then, there exist $\phi \in \Gamma_1$, $\psi \in \Gamma_2$ and $\chi \in \Gamma_3$ such that $\vdash \phi \wedge \psi \to \neg\chi$.*

Proof. See Lemma 7 in HMS. □

Lemma 3. *Let ϕ be a formula with $Pr(\phi) \subseteq \alpha \subseteq X$. Then, the following is a theorem:*

$$a^i_\alpha \phi \leftrightarrow \bigwedge_{x \in Pr(\phi)} a^i_\alpha x.$$

Proof. The proof is effectively the same as the proof of Lemma 1 in HMS. □

Lemma 4. *Let $\omega \in \Omega_\alpha$, $\alpha \subseteq X$. Then ω is closed under inferences in the following sense:*

1. *If ϕ is a theorem such that $\phi \in \mathcal{L}_\alpha$, then $\phi \in \omega$.*
2. *If $\phi \in \omega$ and $\phi \to \psi$ is a theorem such that $\psi \in \mathcal{L}_\alpha$, then $\psi \in \omega$.*
3. *If $\phi_1, \ldots, \phi_n \in \omega$, then $\bigwedge_{i=1}^n \phi_i \in \omega$.*

Proof. See Lemma 4 in HMS and Theorem 2.18 in Chellas [1980]. □

Lemma 5. *If $Pr(\phi) \subseteq \alpha$ then $a^i_\alpha \phi \to k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem.*

Proof. Lemma 3 implies that $a^i_\alpha \phi \to \bigwedge_{x \in Pr(\phi)} a^i_\alpha x$ is a theorem. From Axiom 5 we have that $a^i_\alpha \phi \to k^i_\alpha \phi \vee k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem. Because $k^i_\alpha \phi \to a^i_\alpha \phi$ is a theorem and from Axiom 4, we have that $k^i_\alpha \phi \to \bigvee_{Pr(\phi) \subseteq \beta \subseteq \alpha} k^i_\alpha k^i_\beta \phi$ is a theorem. From Axioms AA and A, we have $k^i_\beta \phi \to a^i_\beta \phi$ and $a^i_\beta \phi \to a^i_{Pr(\phi)} \phi$. From Implication and from RK-Inference we have that $k^i_\alpha k^i_\beta \phi \to k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem. From Implication, $k^i_\alpha \phi \to k^i_\alpha a^i_{Pr(\phi)} \phi$ and therefore $a^i_\alpha \phi \to k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem.

□

Lemma 6. *If ϕ is a theorem and $Pr(\phi) \subseteq \alpha \subseteq X$, then $a^i_\alpha \phi \to k^i_\alpha \phi$ is a theorem.*

Proof. If ϕ is a theorem then, because $\phi \to (a^i_{Pr(\phi)} \phi \to \phi)$ is an instance of a valid formula of PC, by Modus Ponens $a^i_{Pr(\phi)} \phi \to \phi$ is also a theorem. By RK-Inference, $k^i_\alpha a^i_{Pr(\phi)} \phi \to k^i_\alpha \phi$ is a theorem. From Lemma 5, $a^i_\alpha \phi \to k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem. It follows by Implication that $a^i_\alpha \phi \to k^i_\alpha \phi$ is a theorem. □

For every $i \in I$ and $\omega \in \Omega_\alpha$, $\alpha \subseteq X$, define $a(\omega, i) := \{x \in X : a^i_\alpha x \in \omega\}$.

Lemma 7. *For every $\omega \in \Omega$ and $i \in I$, $\omega \cap \mathcal{L}_{a(\omega,i)} \in P^i(\omega)$.*

Proof. See proof of Proposition 5 of HMS, where their Lemma 1 and Lemma 4 correspond to Lemma 3 and Lemma 4 of the present paper, respectively. □

Proof of Proposition 2. Nonemptiness follows from Lemma 7. For 0., by (ii) of the definition of P^i and Lemma 3 we have $P^i(\omega) \subseteq \Omega_{a(\omega,i)}$. Property 1. follows from Lemma 7.

For 2., let $\omega' \in P^i(\omega)$ and $\omega \in \Omega_\alpha$. Lemma 3 and the maximality of ω imply that $\bigwedge_{x\in\alpha'} a^i_\alpha x \bigwedge_{y\in\alpha\setminus\alpha'} \neg a^i_\alpha y \in \omega$, where $\alpha' = a(i, \omega)$. We first show that $P^i(\omega') \subseteq P^i(\omega)$. Suppose that $\omega'' \in P^i(\omega')$ and that $k^i_\alpha \phi \in \omega$. From Axiom 4 and Lemma 4 we have $k^i_\alpha k^i_{\alpha'} \phi \in \omega$. From the definition of P^i we have $k^i_{\alpha'} \phi \in \omega'$, where $\omega' \in \Omega_{\alpha'}$. Hence, $\phi \in \omega''$. If $a^i_\alpha \phi \in \omega$, then from Lemma 5 we have $k^i_\alpha a^i_{Pr(\phi)} \phi \in \omega$, which implies $a^i_{Pr(\phi)} \phi \in \omega'$. Because Axiom AA implies that $a^i_{Pr(\phi)} \phi \rightarrow a^i_{\alpha'} \phi$ is a theorem, we have $a^i_{\alpha'} \phi \in \omega'$ and therefore $\phi \in \omega''$ or $\neg\phi \in \omega''$. Conversely, ($\phi \in \omega''$ or $\neg\phi \in \omega''$) implies $a^i_{\alpha'} \phi \in \omega'$ and hence $a^i_\alpha a^i_{\alpha'} \phi \in \omega$. From Lemma 3 $a^i_\alpha a^i_{\alpha'} \phi \rightarrow a^i_\alpha \phi$ is a theorem and hence $a^i_\alpha \phi \in \omega$.

For the reverse inclusion, suppose that $\omega'' \in P^i(\omega)$ and $\omega' \in \Omega_{\alpha'}$. If $k^i_{\alpha'} \phi \in \omega'$ then $a^i_\alpha k^i_{\alpha'} \phi \in \omega$ and by Lemma 7, ($k^i_{\alpha'} \phi \in \omega$ or $\neg k^i_{\alpha'} \phi \in \omega$). From Axioms PA1 and AA we have $a^i_{\alpha'} \neg k^i_{\alpha'} \phi \in \omega$. Lemmas 3 and 4 imply that $\bigwedge_{x\in\alpha'} a^i_\alpha x \in \omega$. Axiom 5 implies that $a^i_{\alpha'} \neg k^i_{\alpha'} \phi \wedge \neg k^i_{\alpha'} \phi \bigwedge_{x\in\alpha'} a^i_\alpha x \rightarrow k^i_{\alpha'} \neg k^i_{\alpha'} \phi$ is a theorem. If $\neg k^i_{\alpha'} \phi \in \omega$ then we have $k^i_{\alpha'} \neg k^i_{\alpha'} \phi \in \omega$ and from Axiom KA and $\alpha' \subseteq \alpha$ we have $k^i_\alpha \neg k^i_{\alpha'} \phi \in \omega$. But this implies that $\neg k^i_{\alpha'} \phi \in \omega'$, a contradiction to the consistency of ω'. Therefore, $k^i_{\alpha'} \phi \in \omega$ which, from KA, implies $k^i_\alpha \phi \in \omega$ and hence $\phi \in \omega''$. Next, $a^i_{\alpha'} \phi \in \omega'$ implies $a^i_\alpha a^i_{\alpha'} \phi \in \omega$ and therefore $a^i_\alpha \phi \in \omega$ and ($\phi \in \omega''$ or $\neg\phi \in \omega''$). Conversely, ($\phi \in \omega''$ or $\neg\phi \in \omega''$) implies $a^i_\alpha \phi \in \omega$. From Lemma 5, $a^i_\alpha \phi \rightarrow k^i_\alpha a^i_{Pr(\phi)} \phi$ is a theorem. Therefore, $k^i_\alpha a^i_{Pr(\phi)} \phi \in \omega$. But then, $a^i_{Pr(\phi)} \phi \in \omega'$ and from AA, $a^i_{\alpha'} \phi \in \omega'$.

For 3., let $S = \Omega_\alpha$, $S' = \Omega_\beta$, $\alpha \subseteq \beta$. We first show that $k^i_\alpha \phi \in \omega_S = \omega \cap \mathcal{L}_\alpha$ implies $\phi \in \omega_S$. Since $k^i_\alpha \phi \in \omega$ and $k^i_\alpha \phi \rightarrow k^i_\beta \phi$ is a theorem and $k^i_\beta \phi \in \mathcal{L}_\beta$, we have $k^i_\beta \phi \in \omega$ and hence $\phi \in \omega$. Because $k^i_\alpha \phi$ is defined only if $Pr(\phi) \subseteq \alpha$, we have $\phi \in \mathcal{L}_\alpha$ and therefore $\phi \in \omega_S = \omega \cap \mathcal{L}_\alpha$.

Suppose now that $a^i_\alpha \phi \in \omega_S$. Then, $a^i_\alpha \phi \in \omega$, which implies $a^i_\beta \phi \in \omega$ and hence ($\phi \in \omega$ or $\neg\phi \in \omega$). Because $Pr(\phi) \subseteq \alpha$ we have ($\phi \in \omega_S$ or $\neg\phi \in \omega_S$). Conversely, suppose ($\phi \in \omega_S$ or $\neg\phi \in \omega_S$). Then, ($\phi \in \omega$ or $\neg\phi \in \omega$) which implies $a^i_\beta \phi \in \omega$ and as a result $a^i_\alpha \phi \in \omega$. Because $Pr(a^i_\alpha \phi) \subseteq \alpha$ we have $a^i_\alpha \phi \in \mathcal{L}_\alpha$ and therefore $a^i_\alpha \phi \in \omega_S$. As a result, $\omega_S \in P^i(\omega_S)$.

For 4., let $S = \Omega_\alpha$, $\omega \in S' = \Omega_\beta$ and $\alpha \subseteq \beta$. Suppose $\omega' \in P^i(\omega)$ and set $S'' = \Omega_{a(\omega_S,i)}$. We need to show that $\omega'_{S''} \in P(\omega_S)$. Suppose $k^i_\alpha \phi \in \omega \cap \mathcal{L}_\alpha = \omega_S$. Because $k^i_\alpha \phi \rightarrow k^i_\beta \phi$ is a theorem and $k^i_\beta \phi \in \mathcal{L}_\beta$, we have $k^i_\beta \phi \in \omega$ and $\phi \in \omega'$. Moreover, from Axiom A we have that $k^i_\alpha \phi \in \omega_S$ implies $a^i_\alpha \phi \in \omega_S$. From Lemma 3 we have that $Pr(\phi) \subseteq \mathcal{L}_{a(\omega_s,i)}$. Hence, $\phi \in \omega' \cap \mathcal{L}_{a(\omega_s,i)} = \omega'_{S''}$. Suppose $a^i_\alpha \phi \in \omega_S$. Then, $Pr(\phi) \subseteq \mathcal{L}_{a(\omega_s,i)}$ and from Axiom AA we have $a^i_\beta \phi \in \omega$. Therefore $\phi \in \omega'_{S''}$ or $\neg\phi \in \omega'_{S''}$. Conversely, suppose $\phi \in \omega'_{S''}$ or $\neg\phi \in \omega'_{S''}$. Then, $\phi \in \omega'$ or $\neg\phi \in \omega'$, which implies $a^i_\beta \phi \in \omega$ and from Axiom AA, $a^i_\alpha \phi \in \omega_S$.

□

Proof of Lemma 1. The first two claims are obvious. For the third claim, suppose that $\omega \in [k^i_\beta \phi]_{\Omega_\alpha}$. This implies that $k^i_\beta \phi \in \omega$. We need to show that $\omega' = \{\omega\}_{\Omega_\beta} \in K^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})$, or that $P^i(\omega') \subseteq ([\phi]_{\Omega_{Pr(\phi)}})_{\Omega^i(\omega')}$. First, by construction, $k^i_\beta\phi \in \omega' = \omega \cap \mathcal{L}_\beta$. By the definition of P^i, $\omega'' \in P^i(\omega') \subseteq \Omega^i(\omega')$ implies $\phi \in \omega''$. Hence, $\phi \in \{\omega''\}_{\Omega_{Pr(\phi)}} = \omega'' \cap \mathcal{L}_{Pr(\phi)}$, $\{\omega''\}_{\Omega_{Pr(\phi)}} \in [\phi]_{\Omega_{Pr(\phi)}}$ and $\omega'' \in ([\phi]_{\Omega_{Pr(\phi)}})_{\Omega^i(\omega')}$.

For the other direction, suppose that $\omega \in \left(K^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})\right)_{\Omega_\alpha}$, so that $\omega' = \{\omega\}_{\Omega_\beta} \in K^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})$. Hence, $P^i(\omega') \subseteq ([\phi]_{Pr(\phi)})_{\Omega^i(\omega')} \subseteq [\phi]_{\Omega^i(\omega')}$. Suppose that $k^i_\beta\phi \notin \omega'$. Because ω' is maximally consistent in $\mathcal{L}_\beta$ and $k^i_\beta\phi \in \mathcal{L}_\beta$, we have $\neg k^i_\beta\phi \in \omega'$. From the definition of P^i, maximal consistency, Generalized Reflexivity and $P^i(\omega') \subseteq [\phi]_{\Omega^i(\omega')}$, we have $a^i_\beta\phi \in \omega'$. From Lemma 5, $a^i_\beta\phi \rightarrow k^i_\beta a^i_{Pr(\phi)}\phi$ is a theorem. From Lemma 4, $k^i_\beta a^i_{Pr(\phi)}\phi \in \omega'$. Define $ken^i(\omega') := \{\psi : k^i_\beta\psi \in \omega'\}$, which is closed under conjunctions. Then, $a^i_{Pr(\phi)}\phi \in ken^i(\omega')$.

Suppose that $ken^i(\omega') \cup \{\neg\phi\}$ is inconsistent. From Lemma 2, $\psi \rightarrow \phi$ is a theorem, for some $\psi \in ken^i(\omega')$. Then, also $\psi \wedge a^i_{Pr(\phi)}\phi \rightarrow \phi$ is a theorem in the language of ω'. By RK-Inference, $k^i_\beta\psi \wedge k^i_\beta a^i_{Pr(\phi)}\phi \rightarrow k^i_\beta\phi$ is a theorem and hence $k^i_\beta\phi \in \omega'$, a contradiction to the original hypothesis. Hence, $ken^i(\omega') \cup \{\neg\phi\}$ is consistent. This implies that we can extend it to a maximally consistent ω'' in the sub-language of the elements of $P^i(\omega')$. By the definition of P^i, we have $\omega'' \in P^i(\omega')$, contradicting that $P^i(\omega') \subseteq [\phi]_{\Omega^i(\omega')}$. Hence, $k^i_\beta\phi \in \omega'$ and therefore $k^i_\beta\phi \in \omega$.

For the last claim, suppose that $\omega \in [a^i_\beta\phi]_{\Omega_\alpha}$. This implies that $a^i_\beta\phi \in \omega$. We need to show that $\omega' = \{\omega\}_{\Omega_\beta} \in A^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})$, or that $P^i(\omega') \succeq [\phi]_{\Omega_{Pr(\phi)}}$. By construction, $a^i_\beta\phi \in \omega' = \omega \cap \mathcal{L}_\beta$, which implies $\phi \in \omega''$ or $\neg\phi \in \omega''$ for all $\omega'' \in P^i(\omega')$. Hence, $P^i(\omega') \succeq [\phi]_{\Omega_{Pr(\phi)}}$. For the other direction, suppose that $\omega \in \left(A^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})\right)_{\Omega_\alpha}$, so that $\omega' = \{\omega\}_{\Omega_\beta} \in A^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})$. Then, $P^i(\omega') \succeq [\phi]_{\Omega_{Pr(\phi)}}$, which implies that for all $\omega'' \in P^i(\omega')$, $\phi \in \omega''$ or $\neg\phi \in \omega''$. By the definition of P^i, $a^i_\beta\phi \in \omega'$. Therefore, $a^i_\beta\phi \in \omega$ and $\omega \in [a^i_\beta\phi]_{\Omega_\alpha}$. □

Theorem 2. *Suppose $E, F \preceq S$. Then,*

1. **Subjective Necessitation** *For all $\omega \in S$, $\omega \in K_S(\Omega(\omega))$.*
2. **Generalized Monotonicity** $E_{S(E)\vee S(F)} \subseteq F_{S(E)\vee S(F)}$, $F \preceq E \implies K_S(E) \subseteq K_S(F)$.
3. **Conjunction** $K_S(E) \cap K_S(F) = K_S\left(E_{S(E)\vee S(F)} \cap F_{S(E)\vee S(F)}\right)$.
4. **The Axiom of Knowledge** $K_S(E) \subseteq E_S$.
5. **The Axiom of Transparency** $\omega \in K_S(E) \iff \omega \in K_S(K_{\Omega(\omega)}(E))$.
6. **The Axiom of Wisdom** $\omega \in A_S(E) \cap \neg K_S(E) \iff \omega \in K_S(A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E))$.
7. **Symmetry** $U_S(E) = U_S(\neg E)$.

Proof. See Galanis [2007]. □

Lemma 8. *If $E \preceq S \preceq S'$, then $(K^i_S(E))_{S'} \subseteq K^i_{S'}(E)$ and $A^i_{S'}(E) = (A^i_S(E))_{S'}$.*

Proof. Suppose $\omega \in (K^i_S(E))_{S'}$. Then, $\omega_S \in K^i_S(E)$, which implies that $E \preceq P^i(\omega_S)$ and $P^i(\omega_S) \subseteq E_{\Omega^i(\omega_S)}$. Projections Preserve Ignorance implies that $E \preceq P^i(\omega_S) \preceq P^i(\omega)$ and

$P^i(\omega) \subseteq (P^i(\omega_S))_{\Omega^i(\omega)} \subseteq E_{\Omega^i(\omega)}$. Hence, $\omega \in K^i_{S'}(E)$. Moreover, $(K^i_S(E))_{S'} \subseteq K^i_{S'}(E)$ implies $K^i_S(E) \subseteq (K^i_{S'}(E))_S$.

Suppose $\omega \in A^i_{S'}(E)$ and let $E \subseteq S''$. By Generalized Reflexivity and Stationarity, $\{\omega\}_{\Omega^i(\omega)} \in P^i(\{\omega\}_{\Omega^i(\omega)})$. Because $S'' \preceq \Omega^i(\omega) = \Omega^i(\{\omega\}_{\Omega^i(\omega)})$, Projections Preserve Awareness implies that $\{\omega\}_{S''} \in P^i(\{\omega\}_{S''})$. Hence, $S'' = \Omega^i(\{\omega\}_{S''})$. Because $S'' \preceq S$, Projections Preserve Ignorance implies $S'' = \Omega^i(\{\omega\}_{S''}) \preceq \Omega^i(\{\omega\}_S)$. Therefore, $\{\omega\}_S \in A^i_S(E)$ and $\omega \in (A^i_S(E))_{S'}$. For the other direction, suppose that $\omega \in (A^i_S(E))_{S'}$. Then, $\{\omega\}_S \in A^i_S(E)$ and $S'' \preceq \Omega^i(\{\omega\}_S)$. From Projections Preserve Ignorance, $\Omega^i(\{\omega\}_S) \preceq \Omega^i(\omega)$ and therefore $\omega \in A^i_{S'}(E)$.

□

Lemma 9. *Suppose that $\beta \subseteq \alpha \subseteq C(S)$, $\omega \in S$ and $(\underline{\Sigma}, \omega) \models \bigwedge_{x\in\beta} a^i_\alpha x \bigwedge_{y\in\alpha\setminus\beta} \neg a^i_\alpha y$. Then, we have that $C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)})) = \beta$. Conversely, if $\beta \subseteq C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)}))$ and $\bigwedge_{x\in\beta} a^i_\alpha x$ is defined at ω, then $(\underline{\Sigma}, \omega) \models \bigwedge_{x\in\beta} a^i_\alpha x$.*

Proof. Suppose $(\underline{\Sigma}, \omega) \models \bigwedge_{x\in\beta} a^i_\alpha x \bigwedge_{y\in\alpha\setminus\beta} \neg a^i_\alpha y$. Then, there exists state space S' such that $C(S') = \alpha$. Moreover, we have that $\{\omega\}_{C^{-1}(\alpha)} \in \bigcap_{x\in\beta} \left(A^i_{C^{-1}(\alpha)}(C^{-1}(x))\right)$, which implies $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}(C^{-1}(\beta))$ and $\beta \subseteq C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)}))$.

Suppose that $y \in C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)}))$ and $y \notin \beta$. Because $\Omega^i(\{\omega\}_{C^{-1}(\alpha)}) \preceq C^{-1}(\alpha) = S'$ and $C(S') = \alpha$, we have $y \in \alpha$. Then, $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}(C^{-1}(y)) = A^i_{C^{-1}(\alpha)}([y]_{C^{-1}(y)})$ which implies $(\underline{\Sigma}, \omega) \models a^i_\alpha y$, a contradiction. Hence, $C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)})) = \beta$.

For the other direction, suppose $\beta \subseteq C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)}))$, which implies that $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}(C^{-1}(\beta)) \subseteq \bigcap_{x\in\beta} A^i_{C^{-1}(\alpha)}(C^{-1}(x)) = \bigcap_{x\in\beta} A^i_{C^{-1}(\alpha)}([x]_{C^{-1}(x)})$. Therefore, $(\underline{\Sigma}, \omega) \models \bigwedge_{x\in\beta} a^i_\alpha x$.

□

Lemma 10. *Let $\underline{\Sigma}$ be an unawareness model. Then $\Gamma \models_{\underline{\Sigma}} \phi$ iff for all $\omega \in S^*$, whenever ϕ is defined at ω, we have that $(\underline{\Sigma}, \omega) \models \Gamma$ implies $(\underline{\Sigma}, \omega) \models \phi$.*

Proof. The "only if" is straightforward. For the other direction, suppose that there exists a state space S and a state $\omega \in S$ such that $(\underline{\Sigma}, \omega) \models \Gamma$ and $(\underline{\Sigma}, \omega) \models \neg\phi$. By the definition of $\mapsto$ and $\models$, there exists $\omega^* \in S$ such that $\omega^*_S = \omega$, $(\underline{\Sigma}, \omega^*) \models \Gamma$ and $(\underline{\Sigma}, \omega^*) \models \neg\phi$, a contradiction. □

Proof of Corollary 2. This follows from Corollary 1, Proposition 2 and Lemma 1. □

Proof of Theorem 1. Following the approach of HMS, we prove soundness by showing that:

1. All axioms are valid formulas,
2. the set of valid formulas is closed under RK-Inference, and
3. that for every state in every unawareness model the set of formulas that are true in that state is closed under Modus Ponens.

For PC, it is clear that if ϕ is a substitution instance of a valid formula, then $(\underline{\Sigma}, \omega) \models \phi$ for every unawareness model $\underline{\Sigma}$ and $\omega \in S^*$. Using Lemma 10, ϕ is valid.

For Axiom T, suppose that $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi$, where $\omega \in S$. Then, $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$, which implies, by property 4 of Theorem 2 that $\{\omega\}_{C^{-1}(\alpha)} \in ([\phi]_{C^{-1}(Pr(\phi))})_{C^{-1}(\alpha)} \subseteq [\phi]_{C^{-1}(\alpha)}$. Hence, we have $\omega \in [\phi]_S$ and $(\underline{\Sigma}, \omega) \models \phi$.

For Axiom 4, suppose that $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi \underset{x\in\beta}{\wedge} a^i_\alpha x \underset{y\in\alpha\setminus\beta}{\wedge} \neg a^i_\alpha y$. Then, $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$, which implies, by property 5 of Theorem 2, that $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)} K^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))})$. From Lemma 9, we have $C^{-1}(\beta) = \Omega^i(\{\omega\}_{C^{-1}(\alpha)})$ and $K^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))}) = [k^i_\beta \phi]_{C^{-1}(\beta)}$. Therefore, $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([k^i_\beta \phi]_{C^{-1}(\beta)})$ and $(\underline{\Sigma}, \omega) \models k^i_\alpha k^i_\beta \phi$.

For Axiom 5, suppose $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi \underset{x\in\beta}{\wedge} a^i_\alpha x$, which implies $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$. By property 6 of Theorem 2, $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))}) \bigcup K^i_{C^{-1}(\alpha)}\big(\neg K^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))}) \bigcap A^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))})\big)$. From the proof of Lemma 9 and $(\underline{\Sigma}, \omega) \models \underset{x\in\beta}{\wedge} a^i_\alpha x$ we have $C^{-1}(\beta) \preceq \Omega^i(\{\omega\}_{C^{-1}(\alpha)})$. From Lemma 8 we have that

$$\neg K^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))}) \subseteq \Big(\neg K^i_{C^{-1}(\beta)}([\phi]_{C^{-1}(Pr(\phi))})\Big)_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})},$$

$$A^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))}) \subseteq \Big(A^i_{C^{-1}(\beta)}([\phi]_{C^{-1}(Pr(\phi))})\Big)_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}.$$

From property 2 of Theorem 2 and the definition of K^i we have that

$$K^i_{C^{-1}(\alpha)} \neg K^i_{\Omega^i(\{\omega\}_{C^{-1}(\alpha)})}([\phi]_{C^{-1}(Pr(\phi))}) \subseteq K^i_{C^{-1}(\alpha)} \neg K^i_{C^{-1}(\beta)}([\phi]_{C^{-1}(Pr(\phi))})$$

and similarly for the awareness operator. Combining, we have that $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi \vee k^i_\alpha(\neg k^i_\beta \phi \wedge a^i_\beta \phi)$.

The first propositional awareness axiom follows from property 7 of Theorem 2. For the second propositional awareness Axiom, suppose that $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi \wedge a^i_\alpha \psi$. This is equivalent to having $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))}) \cap A^i_{C^{-1}(\alpha)}([\psi]_{C^{-1}(Pr(\psi))}) = A^i_{C^{-1}(\alpha)}([\phi \wedge \psi]_{C^{-1}(Pr(\phi)\cup Pr(\psi))})$ and $(\underline{\Sigma}, \omega) \models a^i_\alpha(\phi \wedge \psi)$.

For Axiom PA3, suppose that $(\underline{\Sigma}, \omega) \models a^i_\alpha k^j_\beta \phi$. Then, we have that $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([k^j_\beta \phi]_{C^{-1}(\beta)})$, which implies that $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}(C^{-1}(\beta))$. Hence, $\beta \subseteq C(\Omega^i(\{\omega\}_{C^{-1}(\alpha)}))$. Because $a^i_\alpha k^j_\beta \phi$ is defined at ω, so is $\underset{x\in\beta}{\wedge} a^i_\alpha x$. From Lemma 9, $(\underline{\Sigma}, \omega) \models \underset{x\in\beta}{\wedge} a^i_\alpha x$. For the other direction, $(\underline{\Sigma}, \omega) \models \underset{x\in\beta}{\wedge} a^i_\alpha x$ implies that

$$\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}(C^{-1}(\beta)) = A^i_{C^{-1}(\alpha)}(K^j_{C^{-1}(\beta)}([\phi]_{C^{-1}(Pr(\phi))})).$$

Therefore $(\underline{\Sigma}, \omega) \models a^i_\alpha k^j_\beta \phi$. The logic is similar for Axiom PA4.

For Axiom A, suppose that $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi$. Then, we have that $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))}) \subseteq A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$ and $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi$.

For Axiom AA, suppose that $(\underline{\Sigma}, \omega) \models a^i_{\alpha'} \phi$ and $Pr(\phi) \subseteq \alpha \subseteq \alpha'$. Then, we have that $\{\omega\}_{C^{-1}(\alpha')} \in A^i_{C^{-1}(\alpha')}([\phi]_{C^{-1}(Pr(\phi))})$. Because $C^{-1}(Pr(\phi)) \preceq C^{-1}(\alpha) \preceq C^{-1}(\alpha')$ and from

Lemma 8, we have $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$ and if $(\underline{\Sigma}, \omega) \mapsto a^i_\alpha \phi$, then $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi$. The other direction is similar.

For Axiom KA, suppose that $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi$ and $Pr(\phi) \subseteq \alpha \subseteq \alpha'$. Then, we have that $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$. Because $C^{-1}(\alpha) \preceq C^{-1}(\alpha')$, Lemma 8 implies that $\{\omega\}_{C^{-1}(\alpha')} \in K^i_{C^{-1}(\alpha')}([\phi]_{C^{-1}(Pr(\phi))})$ and if $(\underline{\Sigma}, \omega) \mapsto k^i_{\alpha'}\phi$, then $(\underline{\Sigma}, \omega) \models k^i_{\alpha'}\phi$.

For the second claim, suppose that $\phi_1, \phi_2, \ldots, \phi_n$ and ϕ are formulas such that $Pr(\phi) \subseteq \bigcup_{i=1}^n Pr(\phi_i)$ and that $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \phi$ is a valid formula. We want to show that if $\bigcup_{i=1}^n Pr(\phi_i) \subseteq \alpha$ then $k^i_\alpha\phi_1 \wedge k^i_\alpha\phi_2 \wedge \ldots \wedge k^i_\alpha\phi_n \to k^i_\alpha\phi$ is also valid. In particular, we need to show that, for any unawareness model $\underline{\Sigma}$ and any $\omega \in S$, if $(\underline{\Sigma}, \omega) \models k^i_\alpha\phi_1 \wedge k^i_\alpha\phi_2 \wedge \ldots \wedge k^i_\alpha\phi_n$ and $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha\phi$, then $(\underline{\Sigma}, \omega) \models k^i_\alpha\phi$. Suppose $(\underline{\Sigma}, \omega) \models k^i_\alpha\phi_1 \wedge k^i_\alpha\phi_2 \wedge \ldots \wedge k^i_\alpha\phi_n$ and $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha\phi$, where $\omega \in S$. Then, we have $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi_1]_{C^{-1}(Pr(\phi_1))}) \cap K^i_{C^{-1}(\alpha)}([\phi_2]_{C^{-1}(Pr(\phi_2))}) \cap \ldots \cap K^i_{C^{-1}(\alpha)}([\phi_n]_{C^{-1}(Pr(\phi_i))})$. From the definition of $\models$ and property 3 of Theorem 2 we have that $K^i_{C^{-1}(\alpha)}([\phi_1]_{C^{-1}(Pr(\phi_1))}) \cap K^i_{C^{-1}(\alpha)}([\phi_2]_{C^{-1}(Pr(\phi_2))}) \cap \ldots \cap K^i_{C^{-1}(\alpha)}([\phi_n]_{C^{-1}(Pr(\phi_i))}) \subseteq K^i_{C^{-1}(\alpha)}([\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n]_{C^{-1}(\cup_{i=1}^n Pr(\phi_i))})$.

Since $\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n \to \phi$ is defined at ω and it is valid, we have $[\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n]_S \subseteq [\phi]_S$. Because $C^{-1}(\cup_{i=1}^n Pr(\phi_i)) \preceq S$, we also have $[\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n]_{C^{-1}(\cup_{i=1}^n Pr(\phi_i))} \subseteq [\phi]_{C^{-1}(\cup_{i=1}^n Pr(\phi_i))}$. From property 2 of Theorem 2 and the definition of K^i we have $K^i_{C^{-1}(\alpha)}([\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n]_{C^{-1}(\cup_{i=1}^n Pr(\phi_i))}) \subseteq K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(\cup_{i=1}^n Pr(\phi_i))}) \subseteq K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$. Hence, $(\underline{\Sigma}, \omega) \models k^i_\alpha\phi$.

For the third claim, let $\underline{\Sigma}$ be an unawareness model, $\omega \in S \in \mathcal{S}$, $(\underline{\Sigma}, \omega) \models \phi$ and $(\underline{\Sigma}, \omega) \models \phi \to \psi$. We need to show that $(\underline{\Sigma}, \omega) \models \psi$. By the definition of an event, $[\phi]_S$, we have that $\omega \in [\phi]_S$ and $\omega \in [\psi \vee \neg\phi]_S \subseteq [\psi]_S \cup \neg[\phi]_S$. Therefore, $\omega \in [\psi]_S$ and $(\underline{\Sigma}, \omega) \models \psi$.

The proof of completeness is identical to that of HMS, using Corollary 2.

□

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